DAY TWENTY EIGHT

Ellipse

Learning & Revision for the Day

- Concept of Ellipse
- Tangent to an Ellipse
- Normal to an Ellipse
- Auxiliary Circle
- Eccentric Angle of a Point
- Diameter and Conjugate Diameters

Concept of Ellipse

Ellipse is the locus of a point in a plane which moves in such a way that the ratio of its distances from a fixed point (focus) in the same plane to its distance from a fixed straight line (directrix) is always constant, which is always less than unity.

Equations of Ellipse in Standard Form

Different forms of an ellipse and their equations are given below

1. Ellipse of the Form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a$

If the coefficient of x^2 has the larger denominator, then its major axis lies along the X-axis and it is said to be horizontal ellipse as shown below.

Various elements of horizontal ellipse are as follows

(i) Centre, $O(0, 0)$

(ii) Coordinates of the vertices : $A(a, 0)$ and $A_1(-a, 0)$

(iii) Equation of the major axis, $y=0$

(iv) Equation of the minor axis, $x=0$

(v) Focal distance of a point (x, y) is $a \pm ex$.

(vi) Major axis, $AA_1=2a$, Minor axis, $BB_1=2b$

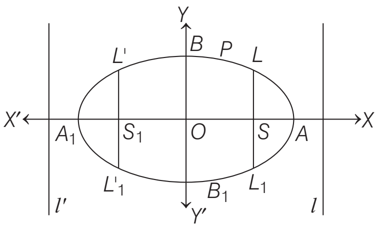
(vii) Foci are $S(ae, 0)$ and $S_1(-ae, 0)$.

(viii) Equations of directrices are $l: x=\frac{a}{e}$, $l': x=-\frac{a}{e}$

(ix) Length of latusrectum, $LL_1=L'L_1'=\frac{2b^2}{a}$

(x) Eccentricity, $e=\sqrt{1-\frac{b^2}{a^2}}$

(xi) Sum of focal distances of a point (x, y) is $2a$.



2. Ellipse of the Form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, 0 < a < b$

If the coefficient of x^2 has the smaller denominator, then its major axis lies along the Y-axis and it is said to be vertical ellipse as shown below.

(i) Centre $O(0, 0)$

(ii) Coordinates of the vertices $B(0, b)$ and $B_1(0, -b)$.

(iii) Equation of the major axis, $x = 0$

(iv) Equation of the minor axis, $y = 0$

(v) Focal distance of a point (x, y) is $b \pm ey$.

(vi) Major axis, $BB_1 = 2b$, Minor axis, $AA_1 = 2a$

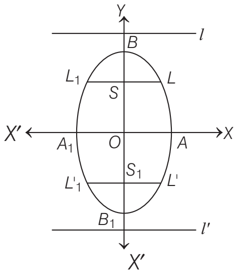
(vii) Foci are $S(0, be)$ and $S_1(0, -be)$.

(viii) Equation of directrices are

$$l : y = \frac{b}{e}, l' : y = -\frac{b}{e}.$$

(ix) Length of latusrectum,

$$LL_1 = L' L_1' = \frac{2a^2}{b}$$



(x) Eccentricity, $e = \sqrt{1 - \frac{a^2}{b^2}}$

(xi) Sum of focal distances of a point (x, y) is $2b$.

Results on Ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$

(i) The equations $x = a\cos\theta$, $y = b\sin\theta$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is the parameter.

(ii) A point (x_1, y_1) with respect to ellipse 'S' lie inside, on or outside the ellipse, if $S_1 < 0, S_1 = 0$ or $S_1 > 0$

where, $S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

(iii) Locus of mid-points of focal chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a > b)$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$.

(iv) Let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be any two points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a > b)$. Then, equation of the chord joining these two points is

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

Tangent to an Ellipse

The equation of tangent to an ellipse for different forms are given below.

(i) In point (x_1, y_1) form, $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$.

(ii) In slope 'm' form, $y = mx \pm \sqrt{a^2m^2+b^2}$. and the point of contact is $\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$

(iii) In parametric $(a\cos\theta, b\sin\theta)$ form, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$.

Results on Tangent to an Ellipse

(i) The line $lx + my + n = 0$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $n^2 = a^2l^2 + b^2m^2$

(ii) The line $y = mx + c$ touches an ellipse, iff $c^2 = a^2m^2 + b^2$ and the point of contact is $\left(\pm\frac{a^2m}{c}, \mp\frac{b^2}{c}\right)$.

(iii) The equation of pair of tangents drawn from an external point $P(x_1, y_1)$ to the ellipse is $SS_1 = T^2$,

where $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1, S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

and $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

(iv) The equation of chord of contact of tangents is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ or $T = 0$.

(v) The equation of chord of an ellipse, whose mid-point is (x_1, y_1), is $T = S_1$, i.e. $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$

(vi) The locus of the point of intersection of perpendicular tangents to the ellipse is a director circle, i.e. $x^2 + y^2 = a^2 + b^2$.

(vii) The point of intersection of the tangents at α and β is

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right).$$

Normal to an Ellipse

The equations of normal in different form to an ellipse are given below

1. In point (x_1, y_1) form, $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

2. In slope 'm' form, $y = mx \pm \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$ at the points $\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{b^2m}{\sqrt{a^2+b^2m^2}}\right)$.

3. In parametric $(a\cos\theta, b\sin\theta)$ form,

$$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2.$$

4. The point of intersection of normals to the ellipse at two points $(a\cos\theta_1, b\sin\theta_1)$ and $(a\cos\theta_2, b\sin\theta_2)$ are (λ, μ),

where $\lambda = \frac{(a^2 - b^2)}{(a)} \cdot \cos\theta_1 \cdot \cos\theta_2 \cdot \frac{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$

and $\mu = -\frac{(a^2 - b^2)}{(b)} \cdot \sin\theta_1 \cdot \sin\theta_2 \cdot \frac{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$

Results on Normal to an Ellipse

(i) The line $lx + my + n = 0$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$

(ii) Four normals can be drawn from a point to an ellipse.

(iii) If the line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2 m^2}$ is the condition of normality of the line to the ellipse.

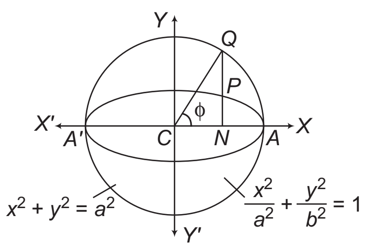
(iv) The points on the ellipse, the normals at which to the ellipse pass through a given point are called **conormal** points.

(v) Tangent at an end of a latusrectum (Ist quadrant) is $\frac{ex}{a} + \frac{\sqrt{1 - e^2}}{b} y = 1$, or $ex + y = a$ and normal is $\frac{ax}{e} - \frac{by}{\sqrt{1 - e^2}} = a^2 - b^2$ or $x - ye = ae^3$

Auxiliary Circle

The circle described on the major axis of an ellipse as diameter is called an auxiliary circle.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, then its auxiliary circle is $x^2 + y^2 = a^2$



Eccentric Angle of a Point

Let P be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PN perpendicular from P on the major axis of the ellipse and produce NP to meet the auxiliary circle in Q. Then, $\angle XCQ = \phi$ is called the eccentric angle of the point P on the ellipse.

So, the coordinates of Q and P are $(a\cos\phi, a\sin\phi)$ and $(a\cos\phi, b\sin\phi)$, where ϕ is an eccentric angle.

Results on Eccentric Angles

(i) Eccentric angles of the extremities of latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $\tan^{-1}\left(\pm\frac{b}{ae}\right)$.

(ii) A circle cut an ellipse in four points real or imaginary. The sum of the eccentric angles of these four concyclic points on the ellipse is an even multiple of π.

(iii) The sum of the eccentric angles of the conormal points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an odd multiple of π.

Diameter and Conjugate Diameters

The locus of the mid-points of a system of parallel chords is called a **diameter.** If $y = mx + c$ represents a system of parallel chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the line $y = -\frac{b^2}{a^2 m} x$ is the equation of the diameter.

The two diameters are said to be **conjugate diameters,** when each bisects all chords parallel to the other.

If $y = mx$ and $y = m_1 x$ be two conjugate diameters of an ellipse, then $m m_1 = -\frac{b^2}{a^2}$

Results on Conjugate Diameters

(i) The area of a parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.

(ii) The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse, i.e. $CP^2 + CD^2 = a^2 + b^2$.

(iii) The product of the focal distance of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

Important Points

(i) The eccentric angles of the ends of a pair of conjugate diameter of an ellipse differ by a right angle.

(ii) The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.

Conjugate Points Two points P and Q are conjugate points with respect to an ellipse, if the polar of P passes through Q and the polar of Q passes through P.

Conjugate Lines Two lines are said to be conjugate lines with respect to an ellipse, if each passes through the pole of the polar.

Pole and Polar Let P be a point inside or outside an ellipse. Then, the locus of the point of intersection of tangents to the ellipse at the point, where secants drawn through 'P' intersect the ellipse is called the **polar** of point P with respect to the ellipse and the point P is called the **pole** of the polar. The polar of a point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 The equation $\frac{x^2}{8-a}+\frac{y^2}{a-2}=1$ will represent an ellipse if

(a) $a \in (1, 4)$ (b) $a \in (-\infty, 2) \cup (8, \infty)$
(c) $a \in (2, 8)$ (d) None of these

2 Equation of ellipse whose minor axis is equal to the distance between the foci and whose latusrectum is 10, is given by (take origin as centre and major axis along X-axis)

(a) $2x^2+y^2=100$ (b) $x^2+2y^2=100$
(c) $2x^2+y^2=50$ (d) None of these

3 Let AB be a rod of length 4 units with A on x and B on Y-axis. Rod AB slides on axes. If point P divides AB in the ratio 1 : 2, locus of P is an ellipse. The eccentricity of this ellipse is

(a) $\frac{3}{4}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{3}$ (d) None

4 Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$ is

(a) $5x^2+3y^2-48=0$ (b) $3x^2+5y^2-15=0$
(c) $5x^2+3y^2-32=0$ (d) $3x^2+5y^2-32=0$

5 If the angle between the straight lines joining foci and the end of minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is 90°, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{\sqrt{2}}$ (d) None

6 The ellipse $4x^2+9y^2=36$ and the straight line $y=mx+c$ intersect in real points only if

(a) $9m^2 \le c^2-4$ (b) $9m^2 > c^2-4$
(c) $9m^2 \ge c^2-4$ (d) None of these

7 If the line $3x+4y=\sqrt{7}$ touches the ellipse $3x^2+4y^2=1$ then the point of contact is

(a) $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right)$ (b) $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$
(c) $\left(\frac{1}{\sqrt{7}}, \frac{-1}{\sqrt{7}}\right)$ (d) None of these

8 The equation of common tangent of the curves $x^2+4y^2=8$ and $y^2=4x$ are

(a) $x-2y+4=0, x+2y+4=0$
(b) $2x-y+4=0, 2x+y+4=0$
(c) $2x-y+2=0, 2x+y+2=0$
(d) None of the above

9 Tangents are drawn to the ellipse $x^2+2y^2=4$ from any arbitrary point on the line $x+y=6$. The corresponding chord of contact will always pass through the fixed point

(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{3}\right)$
(c) $\left(\frac{1}{3}, \frac{1}{3}\right)$ (d) no such fixed point exist.

10 The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is

➔ **JEE Mains 2013**

(a) $x^2+y^2-6y-7=0$ (b) $x^2+y^2-6y+7=0$
(c) $x^2+y^2-6y-5=0$ (d) $x^2+y^2-6y+5=0$

11 If tangent at any point P on the ellipse $7x^2+16y^2=12$ cuts the tangent at the end points of the major axis at the points A and B, then the circle with AB as diameter passes through a fixed point whose coordinates are

(a) $(\pm\sqrt{a^2-b^2}, 0)$ (b) $(\pm\sqrt{a^2+b^2}, 0)$
(c) $(0, \pm\sqrt{a^2-b^2})$ (d) $(0, \pm\sqrt{a^2+b^2})$

12 The length of the common tangent to the ellipse $\frac{x^2}{25}+\frac{y^2}{4}=1$ and the circle $x^2+y^2=16$ intercepted by the coordinate axes is

(a) 5 (b) $2\sqrt{7}$ (c) $\frac{7}{\sqrt{3}}$ (d) $\frac{14}{\sqrt{3}}$

13 The distance of the centre of ellipse $x^2+2y^2-2=0$ to those tangents of the ellipse which are equally inclined from both the axes, is

(a) $\frac{3}{\sqrt{2}}$ (b) $\sqrt{3/2}$ (c) $\sqrt{2}/3$ (d) $\frac{\sqrt{3}}{2}$

14 PQ is a chord of the ellipse through the centre. If the square of its length is the HM of the squares of major and minor axes, find its inclination with X-axis.

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{2\pi}{3}$ (d) None of these

15 If straight line $ax+by=c$ is a normal to the ellipse $4x^2+9y^2=36$, then $4a^2+9b^2$ is equal to

(a) $\frac{169a^2b^2}{c^2}$ (b) $\frac{25a^2b^2}{c^2}$
(c) $\frac{13a^2b^2}{c^2}$ (d) None of these

16 If the normal at the point $P(\theta)$ to the ellipse $5x^2 + 14y^2 = 70$ intersects it again at the point $Q(2\theta)$, then $\cos\theta$ is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

17 An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

(a) $4x^2 + y^2 = 4$ (b) $x^2 + 4y^2 = 8$
(c) $4x^2 + y^2 = 8$ (d) $x^2 + 4y^2 = 16$

18 The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is **→ JEE Mains 2015**

(a) $\frac{27}{4}$ (b) 18 (c) $\frac{27}{2}$ (d) 27

19 The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is **→ JEE Mains 2014**

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

20 Tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ (where, $\theta \in (0, \pi/2)$). Then, the value of θ such that the sum of intercepts on axes made by this tangent is minimum, is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$

21 If the tangent at a point $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the auxiliary circle in two points, the chord joining them subtends a right angle at the centre, then the eccentricity of the ellipse is given by

(a) $(1 + \cos^2\theta)^{-1/2}$ (b) $(1 + \sin^2\theta)$
(c) $(1 + \sin^2\theta)^{-1/2}$ (d) $(1 + \cos^2\theta)$

22 Chord of contact of tangents drawn from the point $P(h, k)$ to the ellipse $x^2 + 4y^2 = 4$ subtends a right angle at the centre. Locus of the point P is

(a) $x^2 + 16y^2 = 20$ (b) $x^2 + 8y^2 = 10$
(c) $16x^2 + y^2 = 20$ (d) None of these

23 Tangent at a point P on the ellipse $x^2 + 4y^2 = 4$ meets the X-axis at B and AP is ordinate of P. If Q is a point on AP produced such that $AQ = AB$, then locus of Q is

(a) $x^2 + xy - 4 = 0$ (b) $x^2 - xy + 4 = 0$
(c) $x^2 + xy - 1 = 0$ (d) $x^2 - xy + 1 = 0$

24 A parabola is drawn whose focus is one of the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where, $a > b$) and whose directrix passes through the other focus and perpendicular to the major axis of the ellipse. Then, the eccentricity of the ellipse for which the latusrectum of the ellipse and the parabola are same, is

(a) $\sqrt{2} - 1$ (b) $2\sqrt{2} + 1$ (c) $\sqrt{2} + 1$ (d) $2\sqrt{2} - 1$

25 At a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ tangent PQ is drawn. If the point Q be at a distance $1/p$ from the point P, where p is distance of the tangent from the origin, then the locus of the point Q is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{1}{a^2b^2}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{1}{a^2b^2}$
(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2b^2}$ (d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2b^2}$

26 If CP and CD are semi-conjugate diameters of an ellipse $\frac{x^2}{14} + \frac{y^2}{8} = 1$, then $CP^2 + CD^2$ is equal to

(a) 20 (b) 22 (c) 24 (d) 26

27 If θ and ϕ are eccentric angles of the ends of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\theta - \phi$ is equal to

(a) $\pm\frac{\pi}{2}$ (b) $\pm\pi$ (c) 0 (d) None of these

28 The coordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the ΔPON is maximum, where O denotes the origin and N, the foot of the perpendicular from O to the tangent at P, is

(a) $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

(b) $\left(\pm\frac{a^2}{\sqrt{a^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2-b^2}}\right)$

(c) $\left(\pm\frac{2a^2}{\sqrt{a^2+b^2}}, \pm\frac{2b^2}{\sqrt{a^2+b^2}}\right)$

(d) $\left(\frac{2a^2}{\sqrt{a^2-b^2}}, \frac{2b^2}{\sqrt{a^2-b^2}}\right)$

29 If α, β, γ are the eccentric angles of three points on the ellipse $4x^2 + 9y^2 = 36$, the normals at which are concurrent, then $\sin(\alpha + \beta) + \sin(\beta + \gamma) + \sin(\gamma + \alpha)$ is equal to

(a) $\frac{2}{3}$ (b) $\frac{4}{9}$ (c) 0 (d) None

30 A triangle is drawn such that it is right angled at the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where, $a > b$) and its other two vertices lie on the ellipse with eccentric angles α and β. Then, $\frac{1 - e^2\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$ is equal to

(a) $\frac{a^2}{a^2+b^2}$ (b) $\frac{a^2+b^2}{a^2}$ (c) $\frac{a^2}{a^2-b^2}$ (d) $\frac{a^2-b^2}{a^2}$

Directions (Q. Nos. 31-35) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes* (a), (b), (c) and (d) *given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

31 Let $S_1 \equiv (x-1)^2 + (y-2)^2 = 0$ and $S_2 \equiv (x+2)^2 + (y-1)^2 = 0$ be the equations of two circles.

Statement I Locus of centre of a variable circle touching two circles S_1 and S_2 is an ellipse.

Statement II If a circle $S_1 = 0$ lies completely inside the circle $S_2 = 0$, then locus of centre of variable circle $S = 0$ which touches both the circles is an ellipse.

32 **Statement I** If $P\left(\frac{3\sqrt{3}}{2}, 1\right)$ is a point on the ellipse $4x^2 + 9y^2 = 36$. Circle drawn AP as diameter touches another circle $x^2 + y^2 = 9$, where $A \equiv (-\sqrt{5}, 0)$.

Statement II Circle drawn with focal radius as diameter touches the auxiliary circle.

33 **Statement I** The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.

Statement II If $y = mx$ and $y = m_1x$ are two conjugate diameters of an ellipse, then $mm_1 = -\frac{b^2}{a^2}$.

34 **Statement I** The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by the line $x + y = b$ is $a^2 + 6ab - 7b^2 \geq 0$.

Statement II Equation of chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid-point is (x_1, y_1), is $T = S_1$.

35 **Statement I** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement II If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 The eccentricity of an ellipse whose centre is at origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is

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(a) $4x + 2y = 7$ (b) $x + 2y = 4$
(c) $2y - x = 2$ (d) $4x - 2y = 1$

2 The locus of the mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$
(c) $x^2 + y^2 = a^2 + b^2$ (d) None of these

3 Foot of normal to the ellipse $4x^2 + 9y^2 = 36$ having slope 2 may be

(a) $\left(\frac{9}{5}, \frac{-8}{5}\right)$ (b) $\left(\frac{9}{5}, \frac{8}{5}\right)$
(c) $\left(\frac{-9}{5}, \frac{8}{5}\right)$ (d) None of these

4 On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are

(a) $\left(\frac{2}{5}, \frac{1}{5}\right), \left(\frac{-2}{5}, \frac{1}{5}\right)$ (b) $\left(\frac{-2}{5}, \frac{1}{5}\right), \left(\frac{2}{5}, \frac{-1}{5}\right)$
(c) $\left(\frac{-3}{5}, \frac{-1}{5}\right), \left(\frac{3}{5}, \frac{1}{5}\right)$ (d) None of these

5 The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectums of the given ellipse at the points

(a) $\left(\pm \frac{3\sqrt{5}}{2}, \pm \frac{2}{7}\right)$ (b) $\left(\pm \frac{3\sqrt{5}}{2}, \pm \frac{\sqrt{19}}{4}\right)$
(c) $\left(\pm 2\sqrt{3}, \pm \frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm \frac{4\sqrt{3}}{7}\right)$

6 The line passing through the extremity A of the major axis and extremity B of the minor axis of ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at AM and the origin O is

(a) $\frac{31}{10}$ (b) $\frac{29}{10}$ (c) $\frac{21}{10}$ (d) $\frac{27}{10}$

7 The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle alingent with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point $(4, 0)$. Then the equation of the ellipse is

(a) $x^2 + 12y^2 = 16$ (b) $4x^2 + 48y^2 = 48$
(c) $4x^2 + 64y^2 = 48$ (d) $x^2 + 16y^2 = 16$

8 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the end points of the latusrectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latusrectum PQ are

(a) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ (b) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$
(c) both (a) and (b) (d) None

9 The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the co-ordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

10 If the line $px + qy = r$ intersects the ellipse $x^2 + 4y^2 = 4$ in points whose eccentric angles differ by $\frac{\pi}{3}$, then r^2 is equal to

(a) $\frac{3}{4}(4p^2 + q^2)$ (b) $\frac{4}{3}(4p^2 + q^2)$
(c) $\frac{2}{3}(4p^2 + q^2)$ (d) None of these

11 The sum of the square of the reciprocals of two perpendicular diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) $\frac{1}{4}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]$ (b) $\frac{1}{2}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]$
(c) $\frac{1}{a^2} + \frac{1}{b^2}$ (d) None of these

12 The area of a triangle inscribed in an ellipse bears a constant ratio of the area of triangle formed by joining the corresponding points on the auxiliary circle of the vertices of the first triangle. This ratio is

(a) $\frac{a}{b}$ (b) $\frac{b}{a}$ (c) $\frac{a^2}{b^2}$ (d) $\frac{b^2}{a^2}$

13 If α, β are the eccentric angles of the extremities of a focal chord of the ellipse $16x^2 + 25y^2 = 400$, then $\tan\left(\frac{\alpha}{2}\right) \tan\left(\frac{\beta}{2}\right)$ is equal to

(a) -4 (b) $\frac{-1}{4}$ (c) $\frac{-3}{8}$ (d) None

14 The tangent and normal drawn to the ellipse $x^2 + 4y^2 = 4$ at the point $P(\theta)$ meets the X-axis at the points A and B. If $AB = 2$, then $\cos^2\theta$ is equal to

(a) $\frac{4}{9}$ (b) $\frac{8}{9}$ (c) $\frac{2}{9}$ (d) None

15 Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ with foci at S and S' and vertices at A and A'. A tangent is drawn at any point P on the ellipse and let R, R', B, B' respectively be the feet of the perpendiculars drawn from S, S', A, A' on the tangent at P. Then, the ratio of the areas of the quadrilaterals $S'R'RS$ and $A'B'BA$ is

(a) $e : 2$ (b) $e : 3$
(c) $e : 1$ (d) $e : 4$

ANSWERS

SESSION 1	**1** (d)	**2** (b)	**3** (b)	**4** (d)	**5** (c)	**6** (c)	**7** (a)	**8** (a)	**9** (b)	**10** (a)
	11 (a)	**12** (d)	**13** (b)	**14** (a)	**15** (b)	**16** (b)	**17** (d)	**18** (d)	**19** (c)	**20** (b)
	21 (c)	**22** (a)	**23** (a)	**24** (a)	**25** (a)	**26** (b)	**27** (a)	**28** (a)	**29** (c)	**30** (b)
	31 (d)	**32** (a)	**33** (b)	**34** (a)	**35** (b)					
SESSION 2	**1** (d)	**2** (a)	**3** (b)	**4** (b)	**5** (c)	**6** (d)	**7** (a)	**8** (c)	**9** (c)	**10** (a)
	11 (a)	**12** (b)	**13** (b)	**14** (a)	**15** (c)					

Hints and Explanations

SESSION 1

1 $\frac{x^2}{8-a}+\frac{y^2}{a-2}=1$ will represent an ellipse if

$8-a>0,\ a-2>0,\ 8-a\neq a-2$

$\Rightarrow\ 2<a<8,\ a\neq 5$

$\Rightarrow\ a\in(2,8)-\{5\}$

2 $2b=2ae$

$\Rightarrow\quad a^2e^2=b^2=a^2(1-e^2)$

$\Rightarrow\quad e^2=\frac{1}{2}$

$\frac{2b^2}{a}=10\Rightarrow b^2=5a$

$\Rightarrow\ a^2(1-1/2)=5a$

$\Rightarrow\quad a=10$

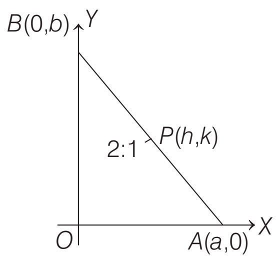
$\therefore\quad b^2=50,\ a^2=100$

Hence, equation of ellipse is

$\frac{x^2}{100}+\frac{y^2}{50}=1$

$\Rightarrow\quad x^2+2y^2=100$

3 Let $A=(a,0),\ B(0,b)$.



P divides AB in the ratio 1:2.

$\therefore\quad h=\frac{2a}{3},\ k=\frac{b}{3}$

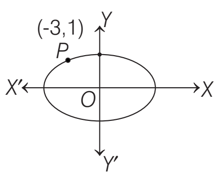
$\therefore\ a^2+b^2=\frac{9h^2}{4}+9k^2=16$ $\quad[\because AB=4]$

$\Rightarrow$ Locus of P is $\frac{x^2}{\left(\frac{64}{9}\right)}+\frac{y^2}{\left(\frac{16}{9}\right)}=1$

$\therefore\ \frac{16}{9}=\frac{64}{9}(1-e^2)\Rightarrow e^2=\frac{3}{4}$ i.e. $e=\frac{\sqrt{3}}{2}$

4 Let the equation of ellipse is

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b)$



It passes through $p(-3,1)$ and $e=\sqrt{\frac{2}{5}}$.

$\therefore\quad \frac{9}{a^2}+\frac{1}{b^2}=1$ ...(i)

and $\quad e^2=1-\frac{b^2}{a^2}$

$\Rightarrow\quad \frac{2}{5}=1-\frac{b^2}{a^2}\Rightarrow\frac{b^2}{a^2}=\frac{3}{5}$

From Eq. (i), $\frac{9}{a^2}+\frac{5}{3a^2}=1$

$\Rightarrow\quad \frac{27+5}{3a^2}=1$

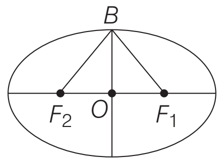
$\Rightarrow\quad a^2=\frac{32}{3}$ and then $b^2=\frac{32}{5}$

$\therefore$ Equation of ellipse is

$\frac{3x^2}{32}+\frac{5y^2}{32}=1\Rightarrow 3x^2+5y^2=32$

5 Let equation of ellipse be

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Then, $F_1=(ae,0),\ F_2=(-ae,0),\ B=(0,b)$.

$\angle F_1BF_2=\frac{\pi}{2}$

$\Rightarrow\quad\left(-\frac{b}{ae}\right)\left(\frac{-b}{-ae}\right)=-1$

$\Rightarrow\quad a^2e^2=b^2=a^2(1-e^2)$

$\Rightarrow\quad e^2=\frac{1}{2}\ \Rightarrow e=\frac{1}{\sqrt{2}}$

6 Solving $4x^2+9y^2=36$ and $y=mx+c$, we get

$4x^2+9(mx+c)^2-36=0$

$\Rightarrow\quad (9m^2+4)x^2+18cmx+9c^2-36=0$

Roots are real, so

$18\times18c^2m^2-4(9m^2+4)(9c^2-36)\geq 0$

$\Rightarrow\quad 9m^2-c^2+4\geq 0$

$\Rightarrow\quad 9m^2\geq c^2-4$

7 Equation of ellipse is $3x^2+4y^2=1$

Tangent at $P(x_1,y_1)$ is

$3xx_1+4yy_1=1$ ...(i)

On comparing Eq. (i) with the given tangent $3x+4y=\sqrt{7}$, $\frac{3x_1}{3}=\frac{4y_1}{4}=\frac{1}{\sqrt{7}}\Rightarrow$

$P(x_1,y_1)=P\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}}\right)$

Clearly, $P\left(\frac{1}{\sqrt{7}},\frac{1}{\sqrt{7}}\right)$ lies on the ellipse, therefore it is a point of contact.

8 Any tangent to parabola $y^2=4x$ is

$y=mx+\frac{1}{m}$

It touches the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$

$\Rightarrow\frac{1}{m^2}=8m^2+2\ [\because c^2=a^2m^2+b^2]$

$\Rightarrow 8m^4+2m^2-1=0$

$\Rightarrow m^2=\frac{1}{4}$ i.e. $m=\pm\frac{1}{2}$

Hence, required tangent are

$x-2y+4=0,\ x+2y+4=0$

9 Any point on the line can be taken as $P(h,6-h)$

Equation of chord of contact of P w.r.t. $x^2+2y^2=4$ is $hx+2(6-h)y=4$ $[T=0]$

$\Rightarrow\quad h(x-2y)+4(3y-1)=0$

which passes through the fixed point $y=\frac{1}{3},\ x=\frac{2}{3}$ i.e., through $\left(\frac{2}{3},\frac{1}{3}\right)$.

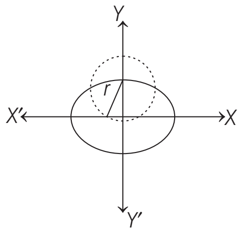
10 Given equation of ellipse is

$\frac{x^2}{16}+\frac{y^2}{9}=1.$

Here, $a=4,\ b=3,\ e=\sqrt{1-\frac{9}{16}}=\frac{\sqrt{7}}{4}$

$\therefore$ Foci is $(\pm\ ae,0)$

$=\left(\pm 4\times\frac{\sqrt{7}}{4},0\right)=(\pm\sqrt{7},0)$



$\therefore\quad$ Radius of the circle,

$r=\sqrt{(ae)^2+b^2}=\sqrt{7+9}$

$=\sqrt{16}=4$

Now, the equation of circle is

$(x-0)^2+(y-3)^2=16$

$\Rightarrow\quad x^2+y^2-6y-7=0$

11 Equation of tangent at any point $P(a\cos\theta,b\sin\theta)$ is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$

The equation of tangents at the end

points of the major axis are
$x = a, x = -a$.

$\therefore$ The intersection point of these tangents are

$$A = \left(a, b\tan\frac{\theta}{2}\right), B = \left(-a, b\cot\frac{\theta}{2}\right).$$

Equation of circle with AB as diameter

$$(x-a)(x+a) + \left(y - b\tan\frac{\theta}{2}\right)\left(y - b\cot\frac{\theta}{2}\right) = 0$$

$$\Rightarrow x^2 - a^2 + y^2 - by\left(\tan\frac{\theta}{2} + \cot\frac{\theta}{2}\right) + b^2 = 0$$

$$\Rightarrow x^2 + y^2 - a^2 + b^2 - 2by\operatorname{cosec}\theta = 0$$

which is the equation of family of circles passing through the point of intersection of the circle

$$x^2 + y^2 - a^2 + b^2 = 0 \quad \text{and} \quad y = 0$$

So, the fixed point is $(\pm\sqrt{a^2 - b^2}, 0)$.

12 The tangent to $\frac{x^2}{25} + \frac{y^2}{4} = 1$ is

$\frac{x}{5}\cos\theta + \frac{y}{2}\sin\theta = 1$. If it is also tangent to the circle, then

$$16 = \frac{1}{\frac{\cos^2\theta}{25} + \frac{\sin^2\theta}{4}} = \frac{100}{4 + 21\sin^2\theta}$$

$$\Rightarrow \sin^2\theta = \frac{3}{28}, \cos^2\theta = \frac{25}{28}$$

If the tangents meets the axes at A and B, then

$$A = \left(\frac{5}{\cos\theta}, 0\right) \text{ and } B = \left(0, \frac{2}{\sin\theta}\right)$$

$$\therefore AB^2 = \frac{25}{\cos^2\theta} + \frac{4}{\sin^2\theta}$$

$$= 28 + \frac{4}{3}\cdot 28 = \frac{196}{3} \Rightarrow AB = \frac{14}{\sqrt{3}}$$

13 Given, equation of ellipse is

$\frac{x^2}{2} + \frac{y^2}{1} = 1$. General equation of tangent to the ellipse of slope m is

$y = mx \pm \sqrt{2m^2 + 1}$

Since, this is equally inclined to axes, so $m = \pm 1$.

Then, tangents are

$$y = \pm x \pm \sqrt{2+1} = \pm x \pm \sqrt{3}$$

Distance of any tangent from origin

$$= \frac{|0 + 0 \pm \sqrt{3}|}{\sqrt{1^2 + 1^2}} = \sqrt{3/2}$$

14 The straight line $x = r\cos\theta, y = r\sin\theta$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where r is given.

$$\therefore \quad \frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad \text{...(i)}$$

But $PQ^2 = 4r^2 = $ HM of $4a^2, 4b^2$

$$\therefore \quad \frac{1}{r^2} = \frac{1}{2a^2} + \frac{1}{2b^2} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} = \frac{1}{2a^2} + \frac{1}{2b^2}$$

It is possible, when $\theta = \frac{\pi}{4}$

15 Given ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 0$, $(a = 3, b = 2)$

Equation of normal is

$3x\sec\theta - 2y\operatorname{cosec}\theta = 9 - 4 = 5$

Comparing it with given normal $ax + by = c$,

$$\frac{3\sec\theta}{a} = \frac{-2\operatorname{cosec}\theta}{b} = \frac{5}{c}$$

$$\Rightarrow \cos\theta = \frac{3c}{5a}, \sin\theta = -\frac{2c}{5b}$$

$$\Rightarrow \frac{9c^2}{25a^2} + \frac{4c^2}{25b^2} = 1$$

$$\Rightarrow 9b^2 + 4a^2 = \frac{25a^2b^2}{c^2}$$

16 Given ellipse is $\frac{x^2}{14} + \frac{y^2}{5} = 1$

$\therefore \quad P(\theta)$ is $(\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$

Equation of normal at $P(\theta)$ is

$$\frac{\sqrt{14}x}{\cos\theta} - \frac{\sqrt{5}y}{\sin\theta} = 14 - 5 = 9$$

$Q(2\theta)$ lies on this normal, therefore

$$\frac{14\cos 2\theta}{\cos\theta} - \frac{5\sin 2\theta}{\sin\theta} = 9$$

$$\Rightarrow 14\cos 2\theta\sin\theta - 10\sin\theta\cos^2\theta = 9\cos\theta\sin\theta$$

$$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow (3\cos\theta + 2)(6\cos\theta - 7) = 0$$

$$\Rightarrow \cos\theta = \frac{-2}{3} \text{ or } \frac{7}{6}$$

$$\Rightarrow \cos\theta = \frac{-2}{3} \text{ as } \cos\theta \neq \frac{7}{6}$$

17 Given,

(i) An ellipse whose semi-minor axis coincides with one of the diameters of the circle $(x-1)^2 + y^2 = 1$.

(ii) The semi-major axis of the ellipse coincides with one of the diameters of circle $x^2 + (y-2)^2 = 4$.

(iii) The centre of the ellipse is at origin.

(iv) The axes of the ellipse are coordinate axes.

Now, diameter of circle $(x-1)^2 + y^2 = 1$ is 2 units and that of circle $x^2 + (y-2)^2 = 4$ is 4 units.

Semi-minor axis of ellipse, $b = 2$ units and semi-major axis of ellipse, $a = 4$ units.

Hence, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\Rightarrow x^2 + 4y^2 = 16$$

18 Area of quadrilateral formed by tangents at the ends of latusrectum $= \frac{2a^2}{e}$.

Here, $a^2 = 9, b^2 = 5, e = \frac{2}{3}$

$$\therefore \text{ Area} = 2 \times 9 \times \frac{3}{2} = 27$$

19 $x^2 + 3y^2 = 6$

$$\Rightarrow \frac{x^2}{6} + \frac{y^2}{2} = 1 \; (a^2 = 6, b^2 = 2) \quad \text{...(i)}$$

Equation of any tangent to Eqs. (i) is

$$y = mx \pm \sqrt{6m^2 + 2} \quad \text{...(ii)}$$

Equation of perpendicular line drawn from centre $(0, 0)$ to Eqs. (ii) is

$$y = -\frac{1}{m}x \quad \text{...(iii)}$$

Eliminating m from Eqs. (ii) and (iii), required locus of foot of perpendicular is

$$y = \left(-\frac{x}{y}\right)x \pm \sqrt{\left(6\frac{x^2}{y^2} + 2\right)}$$

$$\Rightarrow (y^2 + x^2)^2 = 6x^2 + 2y^2$$

20 The equation of tangent from the point $(3\sqrt{3}\cos\theta, \sin\theta)$ to the curve is

$$\frac{x\cos\theta}{3\sqrt{3}} + \frac{y\sin\theta}{1} = 1.$$

Thus, sum of intercepts

$= 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta = f(\theta)$ [say]

$$\Rightarrow f'(\theta) = \frac{3\sqrt{3}\sin^3\theta - \cos^3\theta}{\sin^2\theta\cos^2\theta}$$

Put $f'(\theta) = 0$

$$\Rightarrow \sin^3\theta = \frac{1}{3^{3/2}}\cos^3\theta$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

Now, $f''(\theta) > 0$, for $\theta = \pi/6$ [i.e. minimum]

Hence, value of θ is $\frac{\pi}{6}$.

21 Equation of tangent at $(a\cos\theta, b\sin\theta)$ to the ellipse is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ...(i)

The joint equation of the lines joining the points of intersection of Eq. (i) and the auxiliary circle $x^2 + y^2 = a^2$ to the origin, which is the centre of the circle, is $x^2 + y^2 = a^2\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2$

Since, these lines are at right angles.

$$\therefore 1 - a^2\left(\frac{\cos^2\theta}{a^2}\right) + 1 - a^2\left(\frac{\sin^2\theta}{b^2}\right) = 0$$

[$\because$ coefficient of x^2 + coefficient of $y^2 = 0$]

$$\Rightarrow \quad \sin^2\theta\left(1-\frac{a^2}{b^2}\right)+1=0$$

$$\Rightarrow \quad \sin^2\theta\,(b^2-a^2)+b^2=0$$

$$\Rightarrow \quad (1+\sin^2\theta)(a^2e^2)=a^2$$

$$\Rightarrow \quad e=(1+\sin^2\theta)^{-1/2}$$

22 Chord of contact of $P(h, k)$ to the ellipse

$$x^2+4y^2=4 \text{ is}$$

$$hx+4ky-4=0 \qquad \text{...(i)}$$

Making $x^2+4y^2-4=0$ homogeneous with Eq. (i), we get

$$x^2+4y^2-4\left(\frac{hx+4ky}{4}\right)^2=0$$

Chord of contact subtends right angles at the centre, so

coefficient of x^2 + coefficient of $y^2=0$

$$\therefore \quad 1+4-\frac{1}{4}h^2-4k^2=0$$

$\Rightarrow$ locus of $P(h,k)$ is $x^2+16y^2=20$

23 Ellipse is $x^2+4y^2=4$. $P=(2\cos\theta, \sin\theta)$

Equation of tangent at P is

$2\cos\theta x+4\,y\sin\theta=4$

It meets X-axis at B.

$\therefore B=(2\sec\theta, 0)$, $A=(2\cos\theta, 0)$

$$AQ=AB=2\,|\sec\theta-\cos\theta|=\frac{2\sin^2\theta}{\cos\theta}$$

Let $Q=(h,k)$. Then,

$$h=2\cos\theta,\ k=\frac{2\sin^2\theta}{\cos\theta}=\frac{2(1-\cos^2\theta)}{\cos\theta}$$

$$\Rightarrow \quad hk=4\left(\frac{1-h^2}{4}\right)$$

$\therefore$ Locus of Q is $xy=4-x^2$.

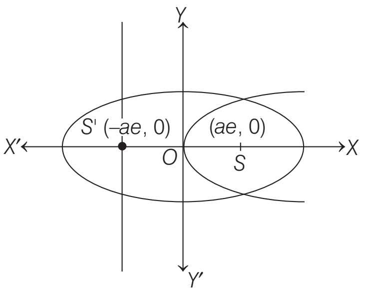
$$\Rightarrow \quad x^2+xy-4=0$$

24 Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Equation of the parabola with focus $S(ae, 0)$ and directrix $x+ae=0$ is $y^2=4aex$.

Now, length of latusrectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

For the two latusrectum to be equal,



$$\frac{2b^2}{a}=4ae \Rightarrow \frac{2a^2(1-e^2)}{a}=4ae$$

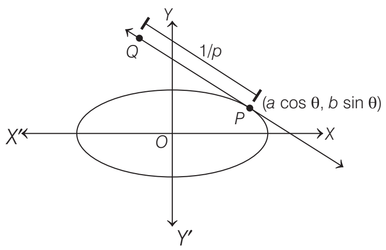
$$\Rightarrow 1-e^2=2e \quad \Rightarrow e^2+2e-1=0$$

Therefore, $e=\frac{-2\pm\sqrt{8}}{2}=-1\pm\sqrt{2}$

Hence, $e=\sqrt{2}-1$ as $0<e<1$ for ellipse.

25 Equation of the tangent at P is

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}$$



The distance of the tangent from the origin is

$$p=\left|\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right|$$

$$\Rightarrow \quad \frac{1}{p}=\frac{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}{ab}$$

Now, the coordinates of the point Q are given as follows,

$$\frac{\frac{x-a\cos\theta}{-a\sin\theta}}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\frac{y-b\sin\theta}{b\cos\theta}}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{1}{p}=\frac{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}{ab}$$

$$\Rightarrow \quad x=a\cos\theta-\frac{a\sin\theta}{ab}$$

$$\text{and} \quad y=b\sin\theta+\frac{b\cos\theta}{ab}$$

$$\Rightarrow \quad \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1+\frac{1}{a^2b^2}$$

is the required locus.

26 CP and CD are semi-conjugate diameters of an ellipse $\frac{x^2}{14}+\frac{y^2}{8}=1$ and let eccentric angle of P is ϕ, then eccentric angle of D is $\frac{\pi}{2}+\phi$, therefore the coordinates of P and D are $(a\cos\phi, b\sin\phi)$ and

$$\left[\sqrt{14}\cos\left(\frac{\pi}{2}+\phi\right), \sqrt{8}\sin\left(\frac{\pi}{2}+\phi\right)\right]$$

i.e. $CP^2+CD^2=(a^2\cos^2\phi+b^2\sin^2\phi)+(a^2\sin^2\phi+b^2\cos^2\phi)$

$$=a^2+b^2=14+8=22$$

27 Let $y=m_1x$ and $y=m_2x$ be a pair of conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ be ends of these two diameters.

Then, $m_1m_2=-\frac{b^2}{a^2}$

$$\Rightarrow \frac{b\sin\theta-0}{a\cos\theta-0}\times\frac{b\sin\phi-0}{a\cos\phi-0}=-\frac{b^2}{a^2}$$

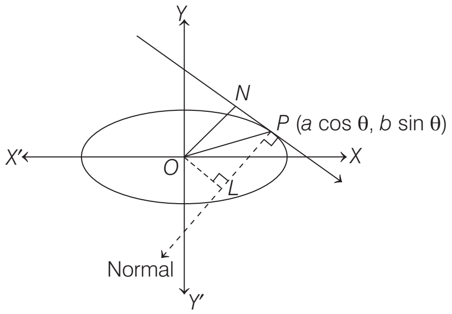
$$\Rightarrow \quad \sin\theta\sin\phi=-\cos\theta\cos\phi$$

$$\Rightarrow \quad \cos(\theta-\phi)=0$$

$$\Rightarrow \quad \theta-\phi=\pm\frac{\pi}{2}$$

28 Equation of tangent at P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$



$$ON=\frac{1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

Equation of the normal at P is $ax\sec\theta-by\,\mathrm{cosec}\,\theta=a^2-b^2$.

$$OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta}}=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

where, L is the foot of perpendicular from O on the normal.

Area of $\Delta PON=\frac{1}{2}\times ON\times OL$ $\quad [\because NP=OL]$

$$=\frac{(a^2-b^2)ab\tan\theta}{a^2\tan^2\theta+b^2}=\frac{(a^2-b^2)ab}{a^2\tan\theta+b^2\cot\theta}$$

which is minimum when $a^2\tan\theta+b^2\cot\theta$ is maximum.

Thus, for area to be minimum, $\tan\theta=\frac{b}{a}$

$$\therefore \cos\theta=\pm\frac{a}{\sqrt{a^2+b^2}},\ \sin\theta=\pm\frac{b}{\sqrt{a^2+b^2}}$$

So, the required coordinates is

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right).$$

29 If $\alpha, \beta, \gamma, \delta$ are eccentric angles of four co-normal points.

Then, $\alpha+\beta+\gamma+\delta=(2n+1)\pi$

and $\Sigma\sin(\alpha+\beta)=0$

Now, $\sin(\alpha+\beta)$

$= \sin[2n\pi + \pi - (y+\delta)] = \sin(\gamma+\delta)$

Similarly, $\sin(\beta+\gamma)=\sin(\alpha+\delta)$

and $\sin(\gamma+\alpha)=\sin(\beta+\delta)$

$\therefore 2[\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)]=0$

30. Equation of the chord with eccentric angles of the extremities as α and β is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

Let $\frac{\alpha+\beta}{2}=\Delta_1$ and $\frac{\alpha-\beta}{2}=\Delta_2$,

so that $\frac{x}{a}\cos\Delta_1+\frac{y}{b}\sin\Delta_1=\cos\Delta_2$

As the triangle is right angled, homogenising the equation of the curve, we get

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{x\cos\Delta_1}{a\cos\Delta_2}+\frac{y\sin\Delta_1}{b\cos\Delta_2}\right)^2=0$$

$$\Rightarrow \left(\frac{1}{a^2}-\frac{\cos^2\Delta_1}{a^2\cos^2\Delta_2}\right)+\left(\frac{1}{b^2}-\frac{\sin^2\Delta_1}{b^2\cos^2\Delta_2}\right)=0$$

[as coefficient of x^2 + coefficient of $y^2=0$]

$\Rightarrow b^2(\cos^2\Delta_2-\cos^2\Delta_1)+a^2(\cos^2\Delta_2-\sin^2\Delta_1)=0$

$\Rightarrow \cos^2\Delta_2(a^2+b^2)-b^2\cos^2\Delta_1-a^2+a^2\cos^2\Delta_1=0$

$\Rightarrow \cos^2\Delta_2(a^2+b^2)=a^2(1-e^2\cos^2\Delta_1)$

$$\Rightarrow \frac{1-e^2\cos^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}=\frac{a^2+b^2}{a^2}$$

31 Let C_1 and C_2 be the centres and R_1 and R_2 be the radii of the two circles. Let $S_1=0$ lie completely inside in the circle $S_2=0$.

Let 'C' and 'r' be the centre and radius of the variable centre.

Then, $CC_2=R_2-r$

and $C_1C=R_1+r$

$\therefore$ $C_1C+C_2C=R_1+R_2$ [constant]

So, the locus of C is an ellipse.

Therefore, Statement II is true.

Hence, Statement I is false (two circles are intersecting).

32 The ellipse is $\frac{x^2}{9}+\frac{y^2}{4}=1$.

So, auxiliary circle is $x^2+y^2=9$ and $(-\sqrt5,0)$ and $(\sqrt5,0)$ are foci.

Hence, Statement I is true, then Statement II is also true.

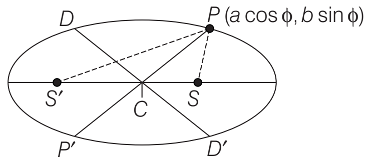
33 Let PCP' and DCD' be the conjugate diameters of an ellipse and let the eccentric angle of P is ϕ, then coordinate of P is $(a\cos\phi, b\sin\phi)$.

So, coordinate of D is $(-a\sin\phi, b\cos\phi)$.

Let S and S' be two foci of the ellipse.

Then, $SP\cdot S'P$

$=(a-ae\cos\phi)\cdot(a+ae\cos\phi)$



$= a^2-a^2e^2\cos^2\phi$ $\left[\begin{array}{l}\Rightarrow b^2=a^2(1-e^2)\\ \Rightarrow a^2-b^2=a^2e^2\end{array}\right]$

$= a^2-(a^2-b^2)\cos^2\phi$

$= a^2\sin^2\phi+b^2\cos^2\phi = CD^2$

34 Let $(t, b-t)$ be a point on the line $x+y=b$, then equation of chord whose mid-point is $(t, b-t)$, is

$$\frac{tx}{2a^2}+\frac{(b-t)y}{2b^2}-1=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}-1 \quad \ldots(i)$$

Since, point $(a,-b)$ lies on Eq. (i), then

$$\frac{ta}{2a^2}-\frac{b(b-t)}{2b^2}=\frac{t^2}{2a^2}+\frac{(b-t)^2}{2b^2}$$

$\Rightarrow t^2(a^2+b^2)-ab(3a+b)t+2a^2b^2=0$

Since, t is real.

$\therefore$ $B^2-4AC\ge 0$

$\Rightarrow a^2b^2(3a+b)^2-4(a^2+b^2)2a^2b^2\ge 0$

$\Rightarrow 9a^2+6ab+b^2-8a^2-8b^2\ge 0$

$\therefore$ $a^2+6ab-7b^2\ge 0$

35. **Statement I** Given, a parabola $y^2=16\sqrt3x$ and an ellipse $2x^2+y^2=4$.

To find the equation of common tangent to the given parabola and the ellipse. This can be very easily done by comparing the standard equation of tangents. Standard equation of tangent with slope 'm' to the parabola $y^2=16\sqrt3x$ is

$$y=mx+\frac{4\sqrt3}{m} \quad \ldots(i)$$

Standard equation of tangent with slope 'm' to the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$ is

$$y=mx\pm\sqrt{2m^2+4} \quad \ldots(ii)$$

If a line L is a common tangent to both parabola and ellipse, then L should be tangent to parabola i.e. its equation should be like Eq. (i) and L should be tangent to ellipse i.e. its equation should be like Eq. (ii) i.e. L must be like both of the Eqs. (i) and (ii).

Hence, comparing Eqs. (i) and (ii), we get

$$\frac{4\sqrt3}{m}=\pm\sqrt{2m^2+4}$$

On squaring both sides, we get

$m^2(2m^2+4)=48$

$\Rightarrow m^4+2m^2-24=0$

$\Rightarrow (m^2+6)(m^2-4)=0$

$\Rightarrow m^2=4$ $[\because m^2\ne -6]$

$\Rightarrow m=\pm 2$

On substituting $m=\pm 2$ in Eq. (i), we get the required equation of the common tangent as

$y=2x+2\sqrt3$ and $y=-2x-2\sqrt3$

Hence, Statement I is correct.

Statement II We have already seen that, if the line $y=mx+\frac{4\sqrt3}{m}$ is a common tangent to the parabola $y^2=16\sqrt3x$ and the ellipse $\frac{x^2}{2}+\frac{y^2}{4}=1$, then it satisfies the equation $m^4+2m^2-24=0$.

Hence, Statement II is also correct but is not able to explain the Statement I. It is an intermediate step in the final answer.

SESSION 2

1 Let equation of ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Equation of directrix is $x=\pm\frac{a}{e}$

$\therefore \frac{a}{e}=4 \Rightarrow a=4e \Rightarrow a=4\times\frac12=2$ $\left[\because e=\frac12\right]$

$e^2=1-\frac{b^2}{a^2}\Rightarrow b^2=3$

$\therefore$ Equation of ellipse $=\frac{x^2}{4}+\frac{y^2}{3}=1$

Equation of normal

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2,\ \frac{4x}{1}-\frac{3y}{\left(\frac32\right)}=1$$

$\left[\because (x_1,y_1)=\left(1,\frac32\right)\right]$

$\Rightarrow$ $4x-2y=1$

2 Let $P(h,k)$ be the mid-point of the focal chord. Then its equation is

$$\frac{hx}{a^2}+\frac{ky}{b^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2} \quad \ldots(i)$$

Eq. (i) is focal chord so it passes through $(ae,0)$

$$\therefore \frac{h\,ea}{a^2}=\frac{h^2}{a^2}+\frac{k^2}{b^2}$$

$\therefore$ Locus of P is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{ex}{a}$

3 Let foot of normal be $P(x_1,y_1)$. Then

slope of normal $=\frac{a^2}{b^2}\frac{y_1}{x_1}=2\Rightarrow 9y_1=8x_1$

P lies on the ellipse $4x^2+9y^2=36$

$\Rightarrow \quad 4x_1^2 + 9\left(\frac{64x_1^2}{81}\right) = 36 \Rightarrow x_1^2 = \frac{81}{25}$

On taking $x_1 = \frac{9}{5}$, we get $y_1 = \frac{8}{5}$

$\therefore$ One possible foot of normal is $\left(\frac{9}{5}, \frac{8}{5}\right)$

4 Slope of tangent $m = \frac{8}{9}$

Let $P(x_1, y_1)$ be the point of contact.
Then, equation of tangent is

$$4xx_1 + 9yy_1 = 1$$

$$\therefore \quad -\frac{4x_1}{9y_1} = \frac{8}{9} \Rightarrow x_1 = -2y_1$$

$$4x_1^2 + 9y_1^2 = 1 \Rightarrow 16y_1^2 + 9y_1^2 = 1$$

$$\Rightarrow \quad y_1 = \pm\frac{1}{5}$$

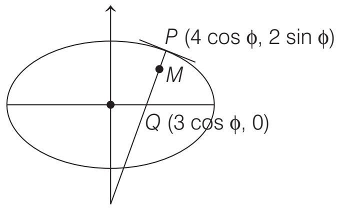
$\therefore$ Required points of contact are

$$\left(\frac{-2}{5}, \frac{1}{5}\right) \text{ and } \left(\frac{2}{5}, \frac{-1}{5}\right)$$

5 Normal is $4x \sec\phi - 2y \operatorname{cosec}\phi = 12$

$Q \equiv (3\cos\phi, 0)$, $M = (\alpha, \beta)$

$$\alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$$



$$\Rightarrow \quad \cos\phi = \frac{2}{7}\alpha \text{ and } \beta = \sin\phi$$

$$\cos^2\phi + \sin^2\phi = 1$$

$$\Rightarrow \quad \frac{4}{49}\alpha^2 + \beta^2 = 1 \Rightarrow \frac{4}{49}x^2 + y^2 = 1$$

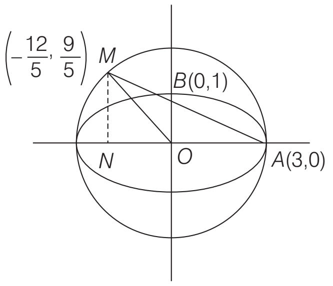
$\Rightarrow \quad$ Latusrectum $x = \pm 2\sqrt{3}$

$$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm\frac{1}{7}\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$$

6 Equation of auxiliary circle is

$$x^2 + y^2 = 9 \qquad \text{...(i)}$$

Equation of AM is $\frac{x}{3} + \frac{y}{1} = 1 \qquad$ ...(ii)



On solving Eqs. (i) and (ii), we get

$$M\left(\frac{-12}{5}, \frac{9}{5}\right)$$

Now, area of ΔAOM

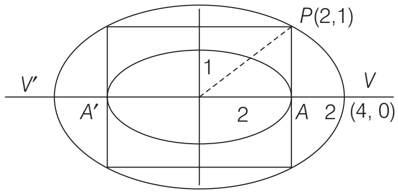
$$= \frac{1}{2}\cdot OA \cdot MN = \frac{27}{10} \text{ sq unit.}$$

7 $x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1$

$\Rightarrow \quad a = 2, b = 1 \Rightarrow P = (2, 1)$

Required Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \quad \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1 \text{ (2, 1) lies on it}$$



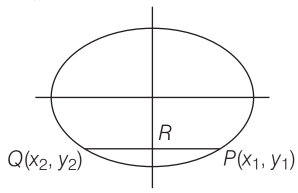
$$\Rightarrow \quad \frac{4}{16} + \frac{1}{b^2} = 1$$

$$\Rightarrow \quad \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$$

$$\therefore \quad \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1$$

$$\Rightarrow \quad x^2 + 12y^2 = 16$$

8 $\frac{x^2}{4} + \frac{y^2}{1} = 1$ and $b^2 = a^2(1 - e^2)$



$$\Rightarrow \quad e = \frac{\sqrt{3}}{2} \Rightarrow P\left(\sqrt{3}, -\frac{1}{2}\right)$$

and $Q = \left(-\sqrt{3}, -\frac{1}{2}\right)$ (given y_1 and y_2 less than 0)

Co-ordinates of mid-point of PQ are

$$R \equiv \left(0, -\frac{1}{2}\right)$$

$PQ = 2\sqrt{3}$ = length of latusrectum.

$\Rightarrow$ Two parabolas are possible whose vertices are $\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $\left(0, \frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

Hence, the equations of the parabolas are $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$ and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

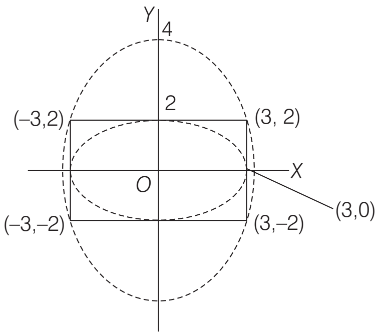
9 Let required ellipse is

$$E_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

It passes through (0, 4)

$$0 + \frac{16}{b^2} = 1 \Rightarrow b^2 = 16$$

It also passes through $(\pm 3, \pm 2)$



$$\frac{9}{a^2} + \frac{4}{b^2} = 1, \frac{9}{a^2} + \frac{1}{4} = 1 \Rightarrow a^2 = 12$$

$$a^2 = b^2(1 - e^2) \Rightarrow \frac{12}{16} = 1 - e^2$$

$$\Rightarrow \quad e^2 = 1 - \frac{12}{16} = \frac{14}{16} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$$

10 Ellipse is $\frac{x^2}{4} + \frac{y^2}{1} = 1$

Let $A(\theta) = (2\cos\theta, \sin\theta)$

Then, $B = \left(2\cos\left(\theta + \frac{\pi}{3}\right), \sin\left(\theta + \frac{\pi}{3}\right)\right)$

Equation of chord AB is

$$\frac{x}{2}\cos\left(\theta + \frac{\pi}{6}\right) + \frac{y}{1}\sin\left(\theta + \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

or given chord is $px + qy = r$

Comparing the coefficient, we get

$$\frac{\cos\left(\theta + \frac{\pi}{6}\right)}{2p} = \frac{\sin\left(\theta + \frac{\pi}{6}\right)}{q} = \frac{\sqrt{3}}{2r}$$

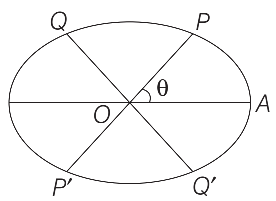
$$\frac{3p^2}{r^2} + \frac{3q^2}{4r^2} = 1$$

$$\Rightarrow \quad 3(4p^2 + q^2) = 4r^2$$

$$\Rightarrow \quad r^2 = \frac{3}{4}(4p^2 + q^2)$$

11 Let POP', QOQ' be two perpendicular diameters.

Let $\angle AOP = \theta$, then $\angle AOQ = \frac{\pi}{2} + \theta$



Then, $P = (OP\cos\theta, OP\sin\theta)$

$$\therefore \quad \frac{OP^2\cos^2\theta}{a^2} + \frac{OP^2\sin^2\theta}{b^2} = 1$$

$$\Rightarrow \quad \frac{1}{OP^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \qquad \text{...(i)}$$

Also $\quad Q = (OQ\cos(90° + \theta), OQ\sin(90° + \theta))$

$$\therefore \quad \frac{1}{OQ^2} = \frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}$$

$$\Rightarrow \quad \frac{1}{OP^2} + \frac{1}{OQ^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$\therefore \frac{1}{(POP')^2} + \frac{1}{(QOQ')^2} = \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

$[\because POP' = 2OP, QOQ' = 2OQ]$

12 Let $P(\theta_1)$, $Q(\theta_2)$, $R(\theta_3)$ be the vertices of ΔPQR, inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then, P', Q', R', corresponding points on auxiliary circle $x^2 + y^2 = a^2$ are $P'(a\cos\theta_1, a\sin\theta_1)$, $Q'(a\cos\theta_2, a\sin\theta_2)$ and $R'(a\cos\theta_3, a\sin\theta_3)$

Δ_1 = Area of ΔPQR

$$= \frac{1}{2}\begin{vmatrix} a\cos\theta_1 & b\sin\theta_1 & 1 \\ a\cos\theta_2 & b\sin\theta_2 & 1 \\ a\cos\theta_3 & b\sin\theta_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}ab\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

Δ_2 = area of $P'Q'R'$

$$= \frac{1}{2}a^2\begin{vmatrix} \cos\theta_1 & \sin\theta_1 & 1 \\ \cos\theta_2 & \sin\theta_2 & 1 \\ \cos\theta_3 & \sin\theta_3 & 1 \end{vmatrix}$$

$$\therefore \quad \frac{\Delta_1}{\Delta_2} = \frac{b}{a} = \text{Constant}$$

13 Equation of chord $P(\alpha)$, $Q(\beta)$ is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right) \quad \text{...(i)}$$

Here, $a = 5, b = 4$. PQ is a focal chord.

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{9}{25} \Rightarrow e = \frac{3}{5}$$

One of the foci $= (ae, 0) = (3, 0)$

Eq. (i) passes through (3, 0)

$$\therefore \quad \frac{3}{5}\cos\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \quad 5\left[\cos\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right] = 3\left[\cos\frac{\alpha}{2}\cos\frac{\beta}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\right]$$

$$\Rightarrow \quad 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2} = -2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$$

$$\Rightarrow \quad \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = -\frac{1}{4}.$$

14 Any point $P(\theta)$ on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ is $P(2\cos\theta, \sin\theta)$

Equation of tangent at $P(\theta)$ is

$$2x\cos\theta + 4y\sin\theta = 4$$

$$\therefore \quad A = (2\sec\theta, 0)$$

Equation of normal of $P(\theta)$ is

$$\frac{2x}{\cos\theta} - \frac{y}{\sin\theta} = 4 - 1 = 3$$

$$\therefore \quad B = \left(\frac{3}{2}\cos\theta, 0\right)$$

$$AB = 2 \Rightarrow \left|2\sec\theta - \frac{3}{2}\cos\theta\right| = 2$$

$$\Rightarrow \quad \left(2 - \frac{3}{2}\cos^2\theta\right)^2 = 4\cos^2\theta$$

$$\Rightarrow \quad 9\cos^4\theta - 40\cos^2\theta + 16 = 0$$

$$\Rightarrow \quad (9\cos^2\theta - 4)(\cos^2\theta - 4) = 0$$

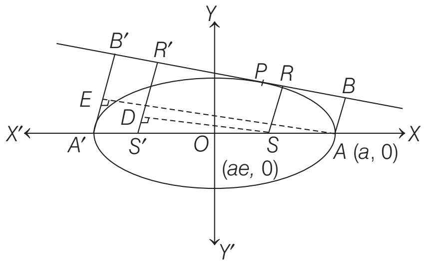
$$\Rightarrow \quad \cos^2\theta = \frac{4}{9} \text{ as } \cos^2\theta \neq 4$$

15 Equation of tangent at P is

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{...(i)}$$

$\therefore$ $S'R'RS$ is a trapezium and its area

$$\Delta_1 = \frac{1}{2}(SR + S'R') \times SD$$



Equation of the line $S'R'$ is

$$y = -\frac{1}{m}(x + ae)$$

$$\Rightarrow \quad x + my + ae = 0 \quad \text{...(ii)}$$

Therefore, $SD = \left|\dfrac{ae + ae}{\sqrt{1+m^2}}\right|$

$$SR = \left|\frac{aem + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$$

and $S'R' = \left|\dfrac{-aem + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$

Then, $\Delta_1 = \frac{1}{2}(S'R' + SR) \times SD$

$$\Rightarrow \quad \Delta_1 = 2ae\left(\frac{\sqrt{a^2m^2 + b^2}}{1+m^2}\right)$$

Area of $A'B'BA$ is

$$\Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$$

Equation of $A'B'$ is

$$y = -\frac{1}{m}(x + a)$$

$$\Rightarrow \quad x + my + a = 0 \quad \text{...(iii)}$$

Therefore, $AE = \left|\dfrac{a + a}{\sqrt{1+m^2}}\right|$

and $AB = \left|\dfrac{ma + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$

and $A'B' = \left|\dfrac{-ma + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$

Then, $\Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$

$$\Rightarrow \quad \Delta_2 = 2a\left(\frac{\sqrt{a^2m^2 + b^2}}{1+m^2}\right)$$

Hence, $\Delta_1 : \Delta_2 = e : 1$